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Liouville–Green–Olver Approximations for Complex Difference Equations*

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Liouville–Green (*LG*) or *WKB* approximations for second-order linear *difference* equations with *complex* coefficients are obtained. Precise *bounds* for the error term in the asymptotic representation of the *LG* recessive solution are given, and the *double asymptotic* nature, with respect to both, *n* and additional parameters, is shown; all this is in the spirit of F. W. J. Olver's rigorous work on the *LG* asymptotics for differential equations. The *holomorphic* character of such error terms, and hence of the *LG* basis, is also established, when the coefficients of the difference equation are holomorphic. Qualitative properties, such as oscillation and growth of the *LG* basis solutions, are displayed. *Second-order* asymptotics with bounds is also obtained, and an application to three-term recurrences satisfied by certain orthogonal polynomials (a subclass of the Blumenthal-Nevai class), is made for illustration. The special case of *ultraspherical* functions of the second kind is worked out in detail. © 1999 Academic Press

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1. INTRODUCTION

Consider the three-term linear homogeneous recurrence

$$Y_{n+2}+A_n Y_{n+1}+B_n Y_n=0, \qquad A_n, B_n \in \mathbf{C}, \quad n \in \mathbf{Z}_\nu, \tag{1}$$

where $\mathbf{Z}_\nu := \{n \in \mathbf{Z}: n \geqslant \nu\}$, ν being a given integer, and assume that the following properties hold:

(i) $A_n \neq 0$ for all $n \in \mathbf{Z}_\nu$;

(ii)

$$\frac{B_n}{A_n A_{n-1}} \to L, \qquad L \in \mathbf{C} \backslash [1/4, +\infty), \qquad L \neq 0, \qquad \text{as} \quad n \to \infty;$$

(iii)

$$\sum_{n=\nu+1}^{\infty} \left| \frac{B_n}{A_n A_{n-1}} - L \right| < \infty.$$

In view of (i), Eq. (1) can be taken into the "canonical form"

$$\Delta^2 y_n + q_n y_n = 0, \qquad n \in \mathbf{Z}_\nu, \tag{2}$$

where Δ denotes the forward difference operator, and

$$q_n := -1 + \frac{4B_n}{A_n A_{n-1}}, \qquad n \geqslant \nu+1, \tag{3}$$

by the transformation

$$Y_n = \alpha_n y_n, \qquad \alpha_n := \alpha_{\nu+1} \prod_{k=\nu}^{n-2} \left(-\frac{A_k}{2} \right), \qquad n \geqslant \nu+2, \tag{4}$$

with α_ν and $\alpha_{\nu+1}$ arbitrary, $\alpha_{\nu+1} \neq 0$; see [15]. It is convenient to write q_n in (3) as

$$q_n = a + g_n, \tag{5}$$

with

$$a := 4L - 1, \qquad g_n := 4 \left(\frac{B_n}{A_n A_{n-1}} - L \right). \tag{6}$$

Under the assumptions (ii)–(iii) above, a Liouville–Green (*LG* for short) or *WKB* type approximation result, namely a certain kind of asymptotic approximation accompanied by precise *bounds* for the *error terms* (in the spirit of F. W. J. Olver's work for differential equations, [14]), was proved by the authors for a basis of solutions to Eq. (2), cf. [15, 16, 18]. Asymptotic results then follow for the original three-term recurrence (1) via the transformation (4). Such results, however, refer to the case $a \in \mathbf{R}$ (while g_n can be complex). The cases $a > 0$ (i.e., $L > 1/4$, oscillatory difference equation), and $a < 0$, $a \neq -1$ (i.e., $L < 1/4$, nonoscillatory or "exponential" case), were treated separately in [15], [18]; when $a = 0$, it is required, in addition, that g_n has finite first (or second) moment [16]. The case $a = -1$ ($L = 0$) could not be included in our analysis, since the unperturbed equation $\Delta^2 y_n + a y_n = 0$ degenerates, cf. [18]. For other recent contributions to the asymptotic analysis of second-order linear difference equations, see, e.g., [3, 7, 8, 22, 23, 24]; a rather complete survey on "discrete" *LG* theories can be found in [20].

In this paper we extend our previous results to the case when a takes *complex* values, obtaining an asymptotic representation of the *LG* basis,

$$Y_n^- = \alpha_n (1 - \sqrt{-a})^n \, [1 + \varepsilon_n], \qquad Y_n^+ \sim \alpha_n (1 + \sqrt{-a})^n, \quad n \to \infty, \tag{7}$$

with the error term, ε_n, bounded as in (12) below. Here is the plan of the paper. In Section 2, the main asymptotic theorem is stated and proved; in particular, the *holomorphic* character of the *LG* basis is established, when the coefficients depend holomorphically on some parameter. We also show the *double asymptotic nature* of the *LG* approximation, with respect to n and the parameter a. In Section 3, qualitative results such as oscillation and growth of the *LG* basis solutions are obtained from the asymptotic representations, and an application is made to the three-term recurrence satisfied by a well-known subclass of the Blumenthal-Nevai orthogonal polynomials (cf. [19]), thus completing earlier achievements due to Ismail, Masson and Saff [8]. In Section 4, the first-order asymptotics in the main theorem above, along with the basic qualitative results derived by Elliott for Jacobi polynomials [5], is used to obtain an asymptotic representation *with bounds* of the *ultraspherical functions* of the *second kind*. In Section 5, finally, a *second-order LG* asymptotic theory is developed and applied, in particular, to ultraspherical functions of the second kind, where precise bounds are obtained again.

2. THE MAIN THEOREM

The main results of this paper consist first of a theorem, which extends asymptotic results of the *LG* type, earlier proved in [Theorem 3.5, 18], for difference equations like (2) with $a \in \mathbf{R}^- \backslash \{-1\}$, to *complex* values of a. In addition, the *analyticity* of the *LG* recessive solution, when a and g_n depend holomorphically on a parameter, z, is established. The latter result plays a role in the theory of orthogonal polynomials, as it will be shown in Section 3.

THEOREM 2.1. *Consider the linear second-order difference equation*

$$\Delta^2 y_n + (a + g_n)\, y_n = 0, \qquad n \in \mathbf{Z}_\nu, \tag{8}$$

with $a \in \mathbf{C} \backslash (0, +\infty)$, $a \neq -1$, *and assume* $\sum_{n=\nu}^{\infty} |g_n| < \infty$. *Then, there exist* $n_1 \in \mathbf{Z}_\nu$ *and two linearly independent solutions to* (8), *respectively recessive and dominant*,

$$y_n^- = (\lambda_-)^n\,[1 + \varepsilon_n], \qquad n \geqslant n_1, \tag{9}$$

$$y_n^+ \sim (\lambda_+)^n, \qquad n \to \infty, \tag{10}$$

where

$$\lambda_\pm := 1 \pm \sqrt{-a} \tag{11}$$

are the roots of the characteristic equation associated to (8) *with* $g_n \equiv 0$, *the square-root denoting the principal branch. For the error term*, ε_n, *the estimate*

$$|\varepsilon_n| \leqslant \frac{V_n}{1 - V_n}, \qquad V_n := \left| \frac{1}{\sqrt{-a}(\sqrt{-a} - 1)} \right| \sum_{j=n}^{\infty} |g_j|, \quad n \geqslant n_1, \tag{12}$$

holds, where

$$n_1 := \min\{n \in \mathbf{Z}_\nu \colon V_n < 1\}. \tag{13}$$

Moreover, assume that $a = a(z)$, *and* $g_n = g_n(z)$, $n \geqslant \nu$, *are holomorphic for* $z \in \Omega \subseteq \mathbf{C}$ (Ω *open connected*), *and that all hypotheses above hold for each* $z \in \Omega$. *If, in addition*,

$$\sum_{n=\nu}^{\infty} |g_n| \qquad \text{converges uniformly in } K, \tag{14}$$

K *being an arbitrary compact subset of* Ω, *then, defining*

$$M_n(K) := \max_{z \in K} V_n(z), \qquad n_1(K) := \min\{n \in \mathbf{Z}_\nu \colon M_n(K) < 1\}, \tag{15}$$

besides the pointwise estimate in (12) (*all quantities depending on* $z \in \Omega$), *the uniform estimate*

$$|\varepsilon_n(z)| \leqslant \frac{M_n(K)}{1 - M_n(K)}, \quad \forall z \in K, \qquad n \geqslant n_1(K), \tag{16}$$

holds. Finally, $\varepsilon_n(z)$*, and hence* $y_n^-(z)$*, are holomorphic in every open bounded subset* $E \subseteq \Omega$*, for each* $n \geqslant n_1(\bar{E})$*, while* $y_n^+(z)$ *is holomorphic in* E *for* $n \geqslant n_2(\bar{E})$*, where*

$$n_2(\bar{E}) := \min\{n \geqslant n_1(\bar{E})\colon y_n^-(z) \neq 0,\ 1 + a(z) + g_n(z) \neq 0,\ \forall z \in \bar{E}\}. \tag{17}$$

Remark 2.2. Both solutions, $y_n^-(z)$ and $y_n^+(z)$, may be extended to lower indices using backwards the difference Eq. (8), for every $z \in K$, up to $n = \min\{\mu(K), n_i(K)\}$, for $i = 1$ or 2, respectively, where $\mu(K) := \min\{n \in \mathbf{Z}_\nu\colon 1 + a(z) + g_n(z) \neq 0,\ \forall z \in K\}$. Clearly, the solutions extended in this way are holomorphic in E for $n \geqslant \min\{\mu(\bar{E}),\ n_i(\bar{E})\}$, for $i = 1$ or 2, respectively. Note that holomorphicity of solutions would be trivial starting from holomorphic initial values, but, in the present problem, no guarantee exists a priori that such conditions are satisfied by the *LG* asymptotic solutions.

Remark 2.3. If we consider the original three-term recurrence Eq. (1), when $A_n = A_n(z)$ and $B_n = B_n(z)$ are holomorphic in Ω, and the series in (iii) converges uniformly on compact subests of Ω, then the pointwise limit $L = L(z)$ in (ii) is holomorphic in Ω, and thus $a(z)$ and $g_n(z)$ in (5), (6), are holomorphic in Ω. Therefore, $Y_n^\pm(z) = \alpha_n(z)\, y_n^\pm(z)$ are holomorphic in E for the same indices as $y_n^\pm(z)$, respectively (cf. also Remark 2.2).

Proof of Theorem 2.1. We consider, for convenience, the general case where a and g_n depend (holomorphically) on the complex parameter z. Clearly, the first part of the theorem (when a and g_n are constant with z) will follow immediately. One first looks for a solution of the form (9), obtaining the difference equation

$$(\lambda_-(z))^2\, \Delta^2 \varepsilon_n(z) + 2\lambda_-(z)(\lambda_-(z) - 1)\, \Delta\varepsilon_n(z) + g_n(z)(1 + \varepsilon_n(z)) = 0, \tag{18}$$

for the error term. Then one can check (pointwise in $z \in \Omega$) that any solution to the “summation equation”, of the Volterra type,

$$\varepsilon_n(z) = \sigma(z) \sum_{j=n}^{\infty} [1 - (\rho(z))^{j-n+1}]\, g_j(z)(1 + \varepsilon_j(z)), \tag{19}$$

where

$$\rho(z) := \frac{\lambda_-(z)}{\lambda_+(z)} = \frac{1 - \sqrt{-a(z)}}{1 + \sqrt{-a(z)}},$$
$$\sigma(z) := \frac{1}{2\lambda_-(z)(\lambda_-(z) - 1)} = \frac{1}{2\sqrt{-a(z)}(\sqrt{-a(z)} - 1)}, \tag{20}$$

also satisfies Eq. (18) (provided that the series in Eq. (19) converges). In fact, it is easily obtained

$$\begin{aligned}\Delta\varepsilon_n(z) &= -\sigma(z)[1-(\rho(z))^{j-n}]\, g_j(z)(1+\varepsilon_j(z))\,|_{j=n} \\ &\quad + \sigma(z)\sum_{j=n}^{\infty}[1-(\rho(z))^{j-n}]\, g_j(z)(1+\varepsilon_j(z)) \\ &\quad - \sigma(z)\sum_{j=n}^{\infty}[1-(\rho(z))^{j-n+1}]\, g_j(z)(1+\varepsilon_j(z)) \\ &= \sigma(z)\sum_{j=n}^{\infty}(\rho(z))^{j-n}(\rho(z)-1)\, g_j(z)(1+\varepsilon_j(z)), \end{aligned} \tag{21}$$

and similarly

$$\Delta^2\varepsilon_n(z) = -\sigma(z)\frac{\rho(z)-1}{\rho(z)}\, g_n(z)(1+\varepsilon_n(z)) - \frac{\rho(z)-1}{\rho(z)}\,\Delta\varepsilon_n(z), \tag{22}$$

from which (18) follows.

Let K be an arbitrary compact subset of Ω, and define recursively the sequence $\{\varepsilon_n^s(z)\}_{s=0}^{\infty}$, $z \in K$, by

$$\begin{aligned}\varepsilon_n^0(z) &:= 0, \\ \varepsilon_n^{s+1}(z) &:= \sigma(z)\sum_{j=n}^{\infty}[1-(\rho(z))^{j-n+1}]\, g_j(z)[1+\varepsilon_j^s(z)], \\ & \qquad s = 0, 1, 2, ..., \end{aligned} \tag{23}$$

for any fixed $n \geqslant n_1(K)$. Observe that the maximum in (15), $M_n(K)$, exists, in view of the continuity of V_n in K, which follows immediately from (12), (14). The latter also implies that $M_n(K) \to 0$ as $n \to \infty$, which, in turn, guarantees that $n_1(K)$ is well-defined. It is also crucial to note that the sequence $M_n(K)$ is nonincreasing. The sequence in (23) is well-defined since each series appearing on the right-hand side converges (uniformly in K). Indeed, it is easily proved by induction on s that $|\varepsilon_n^s(z)| \leqslant C_s = C_s(K)$, for all $n \geqslant n_1(K)$ and $z \in K$, where $C_0 := 0$, and $C_s := M_{n_1}(K)(1 + C_{s-1})$. In fact,

$$\begin{aligned}|\varepsilon_n^{s+1}(z)| &\leqslant (1+C_s)\,|\sigma(z)| \sum_{j=n}^{\infty} |1-(\rho(z))^{j-n+1}|\, |g_j(z)| \\ &\leqslant (1+C_s)\, V_n(z) \leqslant (1+C_s)\, M_n(K) \\ &\leqslant (1+C_s)\, M_{n_1}(K) = C_{s+1}, \end{aligned} \tag{24}$$

since $|\rho(z)| < 1$ (see (20)), and $M_n(K)$ is nonincreasing with n (by (12), (15)). Finally, note that $\varepsilon_n^s(z)$ are holomorphic in any fixed open bounded subset $E \subseteq \Omega$, for $n \geqslant n_1(\bar{E})$, in view of the uniform convergence and using Weierstrass theorem (an inductive argument is also used here, taking $K = \bar{E}$).

Define now

$$\varepsilon_n(z) := \sum_{s=0}^{\infty} [\varepsilon_n^{s+1}(z) - \varepsilon_n^s(z)], \qquad z \in K, \qquad n \geqslant n_1(K). \tag{25}$$

Such a series satisfies the "Weierstrass M-test", since it is immediately proved by induction on s that $|\varepsilon_n^{s+1}(z) - \varepsilon_n^s(z)| \leqslant [V_n(z)]^{s+1} \leqslant (M_n(K))^{s+1}$, $s = 0, 1, 2, \ldots$ In fact, by definition, $|\varepsilon_n^1(z)| \leqslant V_n(z) \leqslant M_n(K)$, and

$$\begin{aligned}|\varepsilon_n^{s+1}(z) - \varepsilon_n^s(z)| &\leqslant |\sigma(z)| \sum_{j=n}^{\infty} |1 - (\rho(z))^{j-n+1}| \, |g_j(z)| \, [V_j(z)]^s \\ &\leqslant 2 \, |\sigma(z)| \, [V_n(z)]^s \sum_{j=n}^{\infty} |g_j(z)| = [V_n(z)]^{s+1}.\end{aligned} \tag{26}$$

Hence, the estimate (16) (and thus (12) with $K = \{z\}$) holds, and (by Weierstrass theorem) $\varepsilon_n(z)$ turns out to be holomorphic in E for $n \geqslant n_1(\bar{E})$. Clearly, $y_n^-(z)$ is also holomorphic in E (for $n \geqslant n_1(\bar{E})$) since $\lambda_-(z)$ is holomorphic in Ω.

Next we prove that $\varepsilon_n(z)$ solves the summation Eq. (19) for each $z \in \Omega$. Writing

$$\begin{aligned}\varepsilon_n(z) &= \varepsilon_n^1(z) + \sum_{s=1}^{\infty} [\varepsilon_n^{s+1}(z) - \varepsilon_n^s(z)] \\ &= \sigma(z) \sum_{j=n}^{\infty} [1 - (\rho(z))^{j-n+1}] \, g_j(z) \\ &\quad + \sigma(z) \textstyle\sum_{s=1}^{\infty} \sum_{j=n}^{\infty} [1 - (\rho(z))^{j-n+1}] \, g_j(z) [\varepsilon_j^s(z) - \varepsilon_j^{s-1}(z)],\end{aligned} \tag{27}$$

it suffices to show that interchanging the order of summation is permissible. In fact, the discrete version of the dominated convergence theorem can be applied, in view of the estimate

$$\begin{aligned}&\left| \sum_{s=1}^{S} [1 - (\rho(z))^{j-n+1}] \, g_j(z) [\varepsilon_n^s(z) - \varepsilon_n^{s-1}(z)] \right| \\ &\quad \leqslant 2 \, |g_j(z)| \sum_{s=1}^{S} |\varepsilon_n^s(z) - \varepsilon_n^{s-1}(z)| \\ &\quad \leqslant 2 \, |g_j(z)| \sum_{s=1}^{\infty} [V_n(z)]^s \\ &\quad \leqslant 2 \, |g_j(z)| \frac{V_n(z)}{1 - V_n(z)}, \qquad S \geqslant 1, \qquad j \geqslant n \geqslant n_1(\{z\}),\end{aligned} \tag{28}$$

and the fact that $\sum^{\infty} |g_j(z)| < \infty$. Uniqueness follows at once from (19) by linearity and the fact that $V_n(z) < 1$ for each $z \in \Omega$ and $n \geqslant n_1(\{z\})$.

We now focus on the second (dominant) solution of (18). Proceeding as in [18], an application of Cesaro's theorem for complex sequences, earlier proved in [17], yields (10) (pointwise in z). Note that $y_n^-(z)$ is recessive and $y_n^+(z)$ is dominant for each $z \in \Omega$, since $|\lambda_-(z)/\lambda_+(z)| < 1$ (cf. (11)). The idea is to look for a dominant solution of the form

$$w_n(z) := y_n^-(z) \sum_{k=n_2(\bar{E})}^{n-1} \frac{C_k(z)}{y_k^-(z)\, y_{k+1}^-(z)} =: y_n^-(z)\, S_n(z), \qquad z \in E, \tag{29}$$

(cf. [9], Sec. 3.5, Thm. 3.9), C_k denoting the Casorati determinant of $y_k^-(z)$ and $w_k(z)$. Now,

$$C_k(z) = C_{n_2(\bar{E})}(z) \prod_{j=n_2(\bar{E})}^{k-1} (1 + a(z) + g_j(z)), \qquad z \in E, \tag{30}$$

$C_{n_2(\bar{E})}(z)$ being a nonvanishing function of z to be chosen later (cf. [4, Lemma 2.13]). Having

$$\frac{w_n(z)}{(\lambda_+(z))^n} \sim \left(\frac{\lambda_+(z)}{\lambda_-(z)}\right)^n S_n(z), \qquad as \quad n \to \infty, \quad z \in E, \tag{31}$$

from (9) and using the aforementioned complex version of Cesaro's theorem (pointwise in z), we get

$$\begin{aligned} \frac{w_n(z)}{(\lambda_+(z))^n} &\sim \frac{\Delta S_n(z)}{\Delta(\lambda_+(z)/\lambda_-(z))^n} \\ &= \frac{C_n(z)/(y_n^-(z)\, y_{n+1}^-(z))}{(\lambda_+(z)/\lambda_-(z))^{n+1} - (\lambda_+(z)/\lambda_-(z))^n} \\ &\sim \frac{C_n(z)/(\lambda_-(z))^{2n+1}}{(\lambda_+(z)/\lambda_-(z))^{n+1} - (\lambda_+(z)/\lambda_-(z))^n} \\ &= \frac{C_n(z)}{(\lambda_+(z)\, \lambda_-(z))^n\, (\lambda_+(z) - \lambda_-(z))} \\ &= \frac{C_{n_2(\bar{E})}(z)}{2\sqrt{-a(z)}\, (1 + a(z))^{n_2(\bar{E})}} \prod_{j=n_2(\bar{E})}^{n-1} \left(1 + \frac{g_j(z)}{1 + a(z)}\right), \\ &\qquad n \to \infty, \qquad z \in E. \end{aligned} \tag{32}$$

Here we used the fact that $\lambda_+\lambda_- = 1+a$ and $\lambda_+ - \lambda_- = 2\sqrt{-a}$. The product in (32) converges, uniformly in $\bar{E}$, to a certain nonvanishing function holomorphic in E, say $c(z)$, as it can be shown by the uniform convergence in $\bar{E}$ of the series in (14). Thus, choosing $C_{n_2(\bar{E})}(z) = 2\sqrt{-a(z)}\,(1+a(z))^{n_2(\bar{E})}/c(z)$, we get $w_n(z) \sim (\lambda_+(z))^n$, and that $C_k(z)$ in (30) is holomorphic, i.e. there is a solution $y_n^+(z) := w_n(z) \sim (\lambda_+(z))^n$ as $n \to \infty$, which is holomorphic in E for $n \geqslant n_2(\bar{E})$. ∎

Remark 2.4. Note that even the mere qualitative asymptotic behavior of y_n^+ as $n \to \infty$ could not be obtained by the classical Poincaré or Perron's theorems [1, 4, 10] (cf. Remark 3.7 in [18]). This has been obtained also in [8], by similar techniques, within the framework of orthogonal polynomials; no bounds for the error terms were determined there, however.

Remark 2.5. The double asymptotic nature, with respect to n, as $n \to \infty$, and to the parameter a, as $a \to \infty$, is clear from the estimate (12). In particular, $V_n = O(a^{-1})$ as $a \to \infty$, $a \in \mathbf{C} \backslash [0, +\infty)$. This fact represents a peculiarity of the Liouville–Green (or WKB) approximation for differential as well as for difference equations (cf., respectively, [14], and [15], [16], [18]). Moreover, $y_n^\pm$ are holomorphic functions of a, for a in any open bounded subset of $\mathbf{C} \backslash ([0, +\infty) \cup \{-1\})$, for n sufficiently large (as can be seen immediately choosing $a(z) \equiv z$, and g_n constant with z, in Theorem 2.1).

3. QUALITATIVE RESULTS

The asymptotic results for the *LG* basis, $y_n^\pm \sim (\lambda_\pm)^n$ (see (9), (10)), allow us to display their qualitative behavior in various subsets of the complex a-plane. Recall that a real sequence, y_n, is termed oscillatory when for every $N \in \mathbf{N}$ there exists $n \geqslant N$ such that $y_n y_{n+1} \leqslant 0$, cf. [19, Appendix 1]; a *complex* sequence will be called *oscillatory* whenever *both* its real and imaginary parts are oscillatory, non-oscillatory when it is not oscillatory (i.e., when at least one of the two sequences, $\operatorname{Re} y_n$ and $\operatorname{Im} y_n$, is not oscillatory).

Setting $a := re^{i\theta}$, we have

$$(\lambda_\pm)^n = (1 \pm \sqrt{-a})^n = \left(1 + r \pm 2\sqrt{r}\sin\frac{\theta}{2}\right)^{n/2} \exp(i\,n\arg\lambda_\pm), \quad (33)$$

from which one can see that $(\lambda_\pm)^n$ oscillates if and only if $\arg\lambda_\pm \neq 0$. When $\arg\lambda_\pm = 0$, we have $\theta = \pi$ $(a<0)$ and $|\lambda_\pm| = 1 \pm \sqrt{r}$. Thus $(\lambda_+)^n$ does not oscillate if and only if $a<0$, while $(\lambda_-)^n$ does not oscillate if and only if $-1<a<0$.

Moreover, $|\lambda_+|^2 = 1 + r + 2\sqrt{r}\sin\theta/2 > 1$ for every $a \neq 0$, which implies that $(\lambda_+)^n$ grows exponentially in absolute value, oscillating whenever $a \in (-\infty, 0)$. The behavior of $(\lambda_-)^n$ depends whether $|\lambda_-|<1$ or $|\lambda_-|>1$, that is whether a lies inside or outside the *cardioid* $\mathscr{C}$ defined by

$$r = 4\sin^2\frac{\theta}{2} = 2(1-\cos\theta) \tag{34}$$

(see Fig. 1 in the Appendix).

Going back to the LG basis,

$$\begin{aligned} y_n^\pm &= |\lambda_\pm|^n \exp(i\, n \arg\lambda_\pm)(1+o(1)) \\ &= |\lambda_\pm|^n \{[(1+o(1))\cos(n\arg\lambda_\pm) + o(1))] \\ &\quad + i[(1+o(1))\sin(n\arg\lambda_\pm) + o(1)]\}, \end{aligned} \tag{35}$$

we can show, as in [19, Appendix 1], that both $\cos(n\arg\lambda_\pm)$ and $\sin(n\arg\lambda_\pm)$ alternate in sign and are bounded away from 0 on suitable subsequences when $\arg\lambda_\pm \neq 0$. It follows that the behavior of $y_n^\pm$ is, for each given value of the parameter a, the same as that of $(\lambda_\pm)^n$, respectively. Indeed, y_n^+ grows exponentially in absolute value, oscillating when $a \in \mathbf{C}\backslash(-\infty, 0]$, while y_n^- decreases exponentially when a belongs to the inside of the cardioid (34), oscillates when $a \notin [-1, 0]$, exhibits oscillations of asymptotically constant amplitude when a is on the cardioid, and grows exponentially oscillating when a is outside the cardioid. Table 1 in the Appendix summarizes the various cases.

We now turn to the behavior of the *LG* basis, $Y_n^\pm$, of the original Eq. (1) (cf. (7)), namely to the oscillation and growth of $Y_n^\pm$. These qualitative properties can be derived from those of $y_n^\pm$ by the transformation (4). We shall not analyze all possible cases, focusing on certain special instances, such as the recurrences satisfied by well-known families of orthogonal polynomials (see below). Throughout we shall assume that the assumptions (i)–(iii) in Section 1 hold. As for the oscillation, consider the case $A_n = \tilde{A}_n(1+\omega_n)$ as $n\to\infty$, $\tilde{A}_n \in \mathbf{R}$, $\sum^\infty |\omega_n| < \infty$, which ensures that $\alpha_n \sim const. \prod^n(-\tilde{A}_k/2)$. When $\tilde{A}_n < 0$ (at least for n sufficiently large), the oscillatory character of $y_n^\pm$ is conserved in $Y_n^\pm$, while when $\tilde{A}_n > 0$ or alternates in sign (for n sufficiently large), non-oscillation may be changed into oscillation but not conversely, unless $L<0$. An example of the former occurrence will be shown below. Concerning the latter, we only need to observe that $\arg\lambda_+\neq\pi$ and $\arg\lambda_-\neq\pi$, unless $L<0$ (recall that $\lambda_\pm = 1\pm\sqrt{1-4L}$ by (6) and (11)). In particular, if A_n has a finite (nonzero) real limit A, then, in view of condition (ii) of Section 1, B_n also has a finite limit, and again what was stated above applies on allowing for the sign of A.

Considering the noteworthy case that $A_n\to A\in\mathbf{C}\backslash\mathbf{R}$, and the limit of B_n is real, both $Y_n^\pm$ turn out to be oscillatory (and thus the transformation (4) takes oscillation as well as non-oscillation in $y_n^\pm$ into oscillation in $Y_n^\pm$). In fact, non-oscillation of $Y_n^\pm$ occurs if and only if $\arg\lambda_\pm=-\arg A$, that is $A\pm\sqrt{A^2-4B}\in\mathbf{R}^+$, which necessarily requires that A and B are simultaneously either real or non-real complex.

As for the growth, when $\prod^\infty|A_k/2|<\infty$, the behavior of $Y_n^\pm$ is unchanged with respect to that of $y_n^\pm$ (that is y_n^+ and Y_n^+ grow exponentially, while y_n^- and Y_n^- both grow or decay exponentially). When $\prod^\infty|A_k/2|$ diverges, such behavior may change. This may happen, for instance, when A_n has a finite (nonzero) limit, but then

$$|Y_n^\pm|\sim|\alpha_n(\lambda_\pm)^n|\sim\left|\frac{A}{2}\left(1\pm\sqrt{1-\frac{4B}{A^2}}\right)\right|^n \tag{36}$$

shows that exponential growth or decay occurs according to whether $|(A/2)(1\pm\sqrt{1-(4B/A^2)})|$ exceeds or is less than 1.

The case of finite limits for A_n and B_n, $A\neq 0$ and $B>0$, respectively, is relevant for the asymptotic theory of orthogonal polynomials. Consider, in fact, the well-known $M(0,1)$ class [13], corresponding to an equation like (1) with

$$\begin{aligned} &A_n=a_n-z, \qquad a_n\to 0 \qquad \text{as} \quad n\to\infty,\\ &B_n=\frac{b_n}{4}, \qquad b_n>0, \qquad b_n\to 1 \qquad \text{as} \quad n\to\infty, \end{aligned} \tag{37}$$

under the additional assumption

$$\sum^\infty(|a_n|+|b_n-1|)<\infty, \tag{38}$$

cf. [8, 19]. The assumption (38) ensures that all hypotheses of Section 1 are satisfied with

$$L = L(z) = \frac{1}{4z^2}, \qquad a(z) = -1 + \frac{1}{z^2},$$
$$g_n = g_n(z) = \frac{b_n}{(a_n - z)(a_{n-1} - z)} - \frac{1}{z^2} \sim \frac{b_n - 1}{z^2} + \frac{a_n + a_{n-1}}{z^3} - \frac{a_n a_{n-1}}{z^4}, \tag{39}$$

and thus the Liouville–Green approximation as in Theorem 2.1 holds.

Concerning oscillation and growth of the *LG* basis $y_n^{\pm}(z)$ for every fixed z, the discussion above carries over considering that the cardioid (34) in the complex a-plane is mapped into a "walnut-shaped" curve in the z-plane, say $\mathcal{N}$, by the transformation $z^2 = 1/(1+a)$ (see Fig. 2 in the Appendix). Such a curve is symmetric with respect to both axes, and intersects the real axis at $z = \pm 1$, and the imaginary axis at $z = \pm i/\sqrt{3}$; observe that the inside [outside] of the cardioid is mapped onto the outside [inside] of $\mathcal{N}$. See Table 2 in the Appendix.

In Table 3 the oscillation and growth of $Y_n^{\pm}(z) = \alpha_n(z) y_n^{\pm}(z)$ are shown: $\mathcal{E}$ denotes the ellipse with foci in $z = \pm 1$ and semiaxes 3/4 and 5/4, on which $|\alpha_n(z)(\lambda_+(z))^n| = 1$. The recessive solution $Y_n^-(z)$ is, instead, exponentially decreasing for every z, $z \neq \pm 1$. Note that, as stated above in the general case, we have now examples where non-oscillation of $y_n^{\pm}$ is changed into oscillation of $Y_n^{\pm}$ (while the converse cannot occur); compare Table 2 with Table 3.

It is also worth noting that the classification of the qualitative properties of $Y_n^{\pm}$ above, could have been obtained by the asymptotic analysis accomplished in [8]; in fact, our basis solutions $Y_n^+(z)$ and $Y_n^-(z)$ coincide, respectively, with $Y_n^{(d)}(z)$ and $Y_n^{(s)}(z)$ in [8, Theorems 2a and 2b].

The orthogonal polynomial, say $P_n(z)$, associated with Eqs. (1), (37), (38), being dominant off the spectrum [6], is asymptotic to $Y_n^+(z)$, as $n \to \infty$, up to a multiplicative z-dependent factor. Consequently, oscillation and growth of $P_n(z)$ in different regions of the complex z-plane can be obtained from the analogous properties of $Y_n^+(z)$. In particular, in the special case of Jacobi polynomials we recover the qualitative properties derived in the well-known results of Elliott in [5]. It may be of interest to observe that Elliott's results were obtained by the *LG*-approximation for the linear second-order *differential* equation satisfied by the Jacobi polynomials, where n is considered a parameter. The double asymptotic nature of such an approximation, in fact, allows us to obtain asymptotics with respect to n, as $n \to \infty$. Moreover, Elliott found in [5], by the same techniques, an asymptotic representation for another linearly independent solution to the Jacobi *differential* equation, which turns out to be, at the same time, a recessive solution to the Jacobi *difference* equation (cf. [6]). In the next section, confined, for simplicity, to the ultraspherical case, matching such a representation with the discrete *LG* asymptotics for

$Y_n^-(z)$, we shall be able to obtain asymptotics (with error bounds) for the so-called ultraspherical functions of the second kind.

4. FIRST-ORDER *LG* ASYMPTOTICS FOR THE ULTRASPHERICAL FUNCTIONS OF THE SECOND KIND

In this section we derive, for the purpose of illustration of the theory developed in Section 2, an asymptotic representation of the *LG*-type (hence equipped with a precise bound for the error term), for the ultraspherical functions of the second kind. Such functions are defined by

$$\Pi_n(z) := \int_{-1}^{1} \frac{(1-t^2)^\alpha P_n^{(\alpha,\alpha)}(t)}{z-t}\,dt, \qquad z \in \mathbf{C}\setminus[-1,1], \qquad \alpha > -1, \tag{40}$$

and are known to be recessive solutions to the corresponding ultraspherical difference Eq. [21], i.e. Eq. (1) with

$$\begin{aligned} A_n = A_n(z) &= -\frac{(2n+2\alpha+3)(n+\alpha+2)\,z}{(n+2)(n+2\alpha+2)}, \\ B_n &= \frac{(n+\alpha+1)^2\,(n+\alpha+2)}{(n+2)(n+2\alpha+2)(n+\alpha+1)}, \end{aligned} \tag{41}$$

while the ultraspherical (or Gegenbauer) polynomials, $P_n^{(\alpha,\alpha)}(z)$, are dominant solutions for these values of z (cf. Section 3).

Since the set of recessive solutions to Eq. (1) forms a one-dimensional vector subspace of the space of all solutions [1, Section 6.3], it follows that

$$\Pi_n(z) = C(z)\; Y_n^-(z), \tag{42}$$

for some n-independent complex function $C(z)$. Upon identification of the coefficient, $C(z)$, the main theorem in Section 2 yields the desired representation for $\Pi_n(z)$. For simplicity, the dependence of $\Pi_n(z)$, $C(z)$, $Y_n^-(z)$ (and of other functions involved) on the parameter α is not displayed.

We first identify the function $C(z)$ for $z = x \in \mathbf{R}$, $x > 1$. On the one hand, we recall Elliott's expansion to the first order [5], i.e.,

$$\Pi_n(z) \sim 2^{2n+3\alpha+3/2} \frac{\Gamma^2(n+\alpha+1)(z^2-1)^{(2\alpha-1)/4}}{\Gamma(2n+2\alpha+2)[z+(z^2-1)^{1/2}]^{n+\alpha+1/2}}, \qquad \text{as} \quad n \to \infty, \tag{43}$$

uniformly valid for $z \in \mathbf{C}$ except in a neighborhood of $[-1, 1]$. On the other hand, the transformation $\alpha_n(z)$ defined in (4) becomes in this case

$$\alpha_n(z) = \prod_{k=1}^{n\text{-}2} \left(\frac{z}{2} \frac{(2k+2\alpha+3)(k+\alpha+2)}{(k+2)(k+2\alpha+2)} \right), \tag{44}$$

where we chose $v=1$ and $a_{v+1}=1$, and thus, since $\lambda_-(z) = 1 - \sqrt{1-1/z^2}$, we obtain after some algebra

$$\begin{aligned} Y_n^-(z) &\sim \alpha_n(z)(\lambda_-(z))^n \\ &= \frac{16 z^{n-2} \Gamma(2\alpha+1)\, \Gamma(n+\alpha+1/2)\, \Gamma(n+\alpha+1)}{n!\, (\alpha+2)(2\alpha+3)\, \Gamma(\alpha+1/2)\, \Gamma(\alpha+1)\, \Gamma(n+2\alpha+1)} \\ &\quad \times \left(1 - \sqrt{1 - \frac{1}{z^2}}\right)^n, \end{aligned} \tag{45}$$

as $n \to \infty$, valid for $z \in \mathbf{C} \backslash [-1, 1]$. Hence we derive, for $x > 1$,

$$\begin{aligned} C(x) &= \lim_{n \to \infty} \frac{\Pi_n(x)}{Y_n^-(x)} \\ &= \left(\frac{2\pi}{e}\right)^{1/2} 2^\alpha \frac{(\alpha+2)(2\alpha+3)\, \Gamma(\alpha+1)\, \Gamma(\alpha+1/2)}{16\, \Gamma(2\alpha+1)} \\ &\quad \times \frac{x(1-1/x^2)^{(2\alpha-1)/4}}{[1+\sqrt{1-1/x^2}]^{\alpha+1/2}}. \end{aligned} \tag{46}$$

Here Stirling's formula has been used. Restriction to real values of z is due to the fact that $\sqrt{1-1/z^2} = \sqrt{z^2-1}/z$ for such values, the square roots denoting principal branches. Now, $C(z)$ is identified by its values for $z = x \in \mathbf{R}$, $x > 1$, as in (46), since it can be shown that it is holomorphic in $\mathbf{C} \backslash [-1, 1]$. This is true in view of (42), $\Pi_n(z)$ being holomorphic there (cf. [5]), and of the fact that $Y_n^-(z)$ is holomorphic and nonzero in every open bounded subset of $\mathbf{C} \backslash [-1, 1]$, say E, for all $n \geqslant n_2(\bar{E})$ (cf. Theorem 2.1). Therefore

$$\begin{aligned} \Pi_n(z) = \left(\frac{2\pi}{e}\right)^{1/2} 2^\alpha \frac{\Gamma(n+\alpha+1/2)\, \Gamma(n+\alpha+1)}{\Gamma(n+1)\, \Gamma(n+2\alpha+1)} \\ \times \frac{(1-1/z^2)^{(2\alpha-1)/4}}{[1+\sqrt{1-1/z^2}]^{\alpha+1/2}} z^{n-1} \left[1 - \sqrt{1 - \frac{1}{z^2}}\right]^n [1+\varepsilon_n(z)], \\ n \geqslant n_1(\{z\}), \end{aligned} \tag{47}$$

(cf. (15)), and

$$|\varepsilon_n(z)| \leqslant \frac{|1-4\alpha^2|}{4(2n+2\alpha+1)\ |z^2\sqrt{1-1/z^2}|\ |\sqrt{1-1/z^2}-1| - |1-4\alpha^2|}, \tag{48}$$

$n \geqslant n_1(\{z\})$. The estimate in (48) has been obtained by (12), since one can compute

$$V_n(z) = \frac{1}{2\ |\sqrt{1-1/z^2}|\ |\sqrt{1-1/z^2}-1|} \sum_{j=n}^{\infty} \left| \frac{1-4\alpha^2}{z^2(2j+2\alpha+3)(2j+2\alpha+1)} \right|$$
$$= \frac{|1-4\alpha^2|}{4\ |z^2\sqrt{1-1/z^2}|\ |\sqrt{1-1/z^2}-1|\ (2n+2\alpha+1)}, \tag{49}$$

summing up explicitly the telescopic series in (49). Moreover, such estimate holds uniformly in every compact subset, K, $K \subset \mathbf{C} \backslash [-1, 1]$, for $n \geqslant n_1(K)$, cf. (15).

Consider now the special family of compact subsets of $\mathbf{C} \backslash [-1, 1]$, say $K(R, \varepsilon, \eta)$, obtained deleting from a large disk centered at $z=0$ with radius R, two small disks, $C(\pm 1, \varepsilon)$, centered at $z = \pm 1$, with radius ε, along with a neighborhood of $[-1, 1]$ of radius η. From the estimate

$$V_n(z) \leqslant \frac{|1-4\alpha^2|}{4(2n+2\alpha+1)} \left(1 + \frac{1}{\sqrt{|1-1/z^2|}}\right), \tag{50}$$

derived from (49), one obtains, after some manipulations,

$$V_n(z) \leqslant \frac{|1-4\alpha^2|}{4(2n+2\alpha+1)} \left(1 + \frac{1}{g(\varepsilon)}\right), \qquad z \in K(R, \varepsilon, \eta), \tag{51}$$

where

$$g(\varepsilon) := \frac{(1+4\varepsilon)^{1/2}-1}{(2[3+2\varepsilon-(1+4\varepsilon)^{1/2}])^{1/2}} = O(\varepsilon), \tag{52}$$

which clearly holds when $R \to +\infty$ and $\eta \to 0^+$, that is *uniformly*, for all z in $\mathbf{C} \backslash (C(1, \varepsilon) \cup C(-1, \varepsilon) \cup [-1, 1])$. Therefore, the final estimate for the error term in (47), $\varepsilon_n(z)$, becomes

$$|\varepsilon_n(z)| \leqslant \frac{|1-4\alpha^2|\ (1+1/g(\varepsilon))}{4(2n+2\alpha+1) - |1-4\alpha^2|\ (1+1/g(\varepsilon))}, \qquad n \geqslant N(\varepsilon), \tag{53}$$

where $N(\varepsilon) := \min\{n: 4(2n+2\alpha+1) > |1-4\alpha^2|\ (1+1/g(\varepsilon))\}$, and this estimate is uniformly valid for $z \in \mathbf{C} \backslash (C(1, \varepsilon) \cup C(-1, \varepsilon) \cup [-1, 1])$.

5. SECOND-ORDER DISCRETE *LG* ASYMPTOTICS

The discrete Liouville–Green–Olver asymptotics, developed in Section 2 for the recessive solution, $y_n^-(z)$, can be improved by going to higher-order approximations. We shall do this by splitting the error term ε_n in (9) as $\varepsilon_n = \xi_n + e_n$, with

$$\xi_n = O(V_n), \qquad e_n = o(V_n), \qquad \text{as} \quad n \to \infty, \tag{54}$$

where ξ_n is an explicitly representable sequence, and e_n can be estimated (cf. [19] for a similar extension in the real oscillatory case). In view of the discrete Liouville–Neumann expansion [14, 19] of the error term in (25) (for simplicity we do not display the possible dependence on z), we have

$$\varepsilon_n = \varepsilon_n^s + r_n^s, \qquad |r_n^s| \leqslant \frac{(V_n)^{s+1}}{1 - V_n}, \quad s = 1, 2, ..., \quad n \geqslant n_1, \tag{55}$$

where $\varepsilon_n^s = O(V_n)$ is recursively defined by (23). The sequence ξ_n could be identified with ε_n^s or with its "dominant part" with respect to $(V_n)^s$ (as $n \to \infty$). Below, we follow such an approach for $s = 1$ (*second-order* theory), for the special class of difference Eqs. (8) with

$$g_n = cn^{-p} + O(n^{-q}), \qquad c \in \mathbf{C}, \qquad n \in \mathbf{Z}_\nu, \tag{56}$$

where $q > p > 1$. From (12) it follows that $V_n = O(n^{1-p})$ and hence $(V_n)^2 = O(n^{2(1-p)})$. Moreover, from (23) for $s = 0$, and (56), obtains

$$\begin{aligned}
\varepsilon_n^1 &= \sigma \sum_{j=n}^{\infty} [1 - \rho^{j-n+1}](cj^{-p} + O(j^{-q})) \\
&= \sigma \left\{ c \sum_{j=n}^{\infty} j^{-p} - c\rho^{1-n} \sum_{j=n}^{\infty} \rho^j j^{-p} \right. \\
&\quad \left. + O\left(\sum_{j=n}^{\infty} j^{-q} \right) - \rho^{1-n} O\left(\sum_{j=n}^{\infty} \rho^j j^{-q} \right) \right\},
\end{aligned} \tag{57}$$

and using well-known asymptotic results for the remainders of the harmonic series, and of the polylogarithmic [11] power series (in ρ) [14, Chapt. 8, Sections 3, 5], we get easily

$$\varepsilon_n^1 = \frac{\sigma c}{p-1} n^{1-p} + O(n^{-\min\{p, q-1, 2p-2\}}), \qquad n \geqslant n_1. \tag{58}$$

Note that the constant implied by the O-term in (58) could be estimated explicitly following the procedure above, provided that a similar estimate for the O-term in (56) is given. Moreover, when a and g_n depend holomorphically on $z \in \Omega \subseteq \mathbf{C}$, the estimate in (58) can be given in any fixed compact subset of Ω, say K, for all $n \geqslant n_1(K)$, cf. (15).

We conclude with an application of the second-order discrete *LG* theory developed so far, to orthogonal polynomials. We first note that such a theory could be applied to the subclass of the $M(0, 1)$ class, characterized by $a_n \equiv 0$, and $b_n = 1 + \beta n^{-p} + O(n^{-q})$, $q > p > 1$, which includes, for instance, the entire Jacobi family (cf. (37)–(39) in Section 3). For simplicity, we work out in detail only the special case of the ultraspherical polynomials, thus refining the first-order asymptotics displayed in Section 4. In this case, it is more convenient to split the first-order error term in (55) in a different way. Setting

$$\varepsilon_n^1(z) := \xi_n(z) + \eta_n(z), \tag{59}$$

with

$$\xi_n(z) = \sigma(z) \sum_{j=n}^{\infty} g_j(z) = \frac{1 - 4\alpha^2}{4(2n + 2\alpha + 1)\sqrt{1 - 1/z^2}\,(\sqrt{1 - 1/z^2} - 1)\, z^2}, \tag{60}$$

and

$$\begin{aligned} \eta_n(z) &:= -\sigma(z)(\rho(z))^{1-n} \sum_{j=n}^{\infty} (\rho(z))^j\, g_j(z) \\ &= \frac{\sigma(z)(\rho(z))^{1-n}}{\rho(z) - 1} \left\{ (\rho(z))^n\, g_n(z) + \rho(z) \sum_{j=n}^{\infty} (\rho(z))^j\, \Delta g_j(z) \right\}, \end{aligned} \tag{61}$$

where summation by parts has been used (see (20) for the definition of ρ and σ), the term η_n can now be estimated, since, using telescopic sums

$$\begin{aligned} &\left| \rho(z) \sum_{j=n}^{\infty} (\rho(z))^j\, \Delta g_j(z) \right| \\ &\quad \leqslant |\rho(z)|^{n+1} \sum_{j=n}^{\infty} |\Delta g_j(z)| \\ &\quad = |\rho(z)|^{n+1} \frac{|1 - 4\alpha^2|}{|z|^2} \\ &\qquad \times \sum_{j=n}^{\infty} \left[\frac{1}{(2j + 2\alpha + 1)(2j + 2\alpha + 3)} - \frac{1}{(2j + 2\alpha + 3)(2j + 2\alpha + 5)} \right] \\ &\quad = |\rho(z)|^{n+1} \frac{|1 - 4\alpha^2|}{|z|^2} \frac{1}{(2n + 2\alpha + 1)(2n + 2\alpha + 3)}, \end{aligned} \tag{62}$$

so that, by $|\rho(z)|<1$, we get

$$|\eta_n(z)| \leqslant \frac{|1-4\alpha^2|\,(|1+\sqrt{1-1/z^2}|+|1-\sqrt{1-1/z^2}|)}{4\,|z^2-1|\,|1+\sqrt{1-1/z^2}|\,(2n+2\alpha+1)(2n+2\alpha+3)}. \tag{63}$$

As for the second-order term, $r_n^1(z)$ in (55), using (50), we obtain the refinement of the asymptotic representation for the ultraspherical function of the second kind given in (47),

$$\varepsilon_n(z)=\xi_n(z)+e_n(z), \tag{64}$$

where $\xi_n(z)$ is explicitly given in (59), and the second-order error term can be bounded as

$$\begin{aligned}|e_n(z)| &\leqslant |\eta_n(z)|+|r_n^1(z)|\\ &\leqslant \frac{|1-4\alpha^2|\,(|1+\sqrt{1-1/z^2}|+|1-\sqrt{1-1/z^2}|)}{4\,|z^2-1|\,|1+\sqrt{1-1/z^2}|\,(2n+2\alpha+1)(2n+2\alpha+3)}\\ &\quad+\frac{(1-4\alpha^2)^2}{4\,|z|\,\sqrt{|z^2-1|}\,|\sqrt{1-1/z^2}-1|\,(2n+2\alpha+1)}\\ &\quad\times\frac{1}{4\,|z|\,\sqrt{|z^2-1|}\,|\sqrt{1-1/z^2}-1|\,(2n+2\alpha+1)-|1-4\alpha^2|},\end{aligned} \tag{65}$$

for $z\in\mathbf{C}\backslash[-1,1]$, $n\geqslant n_1(\{z\})$. A uniform estimate for the second-order error term, $e_n(z)$, can also be obtained, along the lines of the previous section. The result is

$$\begin{aligned}|e_n(z)| \leqslant{}& \frac{|1-4\alpha^2|}{2\varepsilon^2}\,\frac{1}{(2n+2\alpha+1)(2n+2\alpha+3)}\\ &+\frac{1}{4(2n+2\alpha+1)}\,\frac{(1-4\alpha^2)^2\,(1+1/g(\varepsilon))^2}{4(2n+2\alpha+1)-|1-4\alpha^2|\,(1+1/g(\varepsilon))},\end{aligned} \tag{66}$$

uniformly valid for $z\in\mathbf{C}\backslash(C(1,\varepsilon)\cup C(-1,\varepsilon)\cup[-1,1])$ and $n\geqslant N(\varepsilon)$, cf. (52), (53).

APPENDIX

In Tables 1–3, the abbreviations "*osc.*" and "*non-osc.*" denote oscillation and non-oscillation, respectively, while the arrow "↑" denotes (exponential) growth, "↓" (exponential) decay, and "—" asymptotically constant modulus.

TABLE 1

	y_n^+	y_n^-
$a \in \mathbf{C} \backslash \mathbf{R}$		
in $\mathscr{C}$	*osc.* $\uparrow$	*osc.* $\downarrow$
on $\mathscr{C}$	*osc.* $\uparrow$	*osc.* $-$
out $\mathscr{C}$	*osc.* $\uparrow$	*osc.* $\uparrow$
$a > 0$	*osc.* $\uparrow$	*osc.* $\uparrow$
$a \in (-1, 0)$	*non-osc.* $\uparrow$	*non-osc.* $\downarrow$
$a \in (-4, -1)$	*non-osc.* $\uparrow$	*osc.* $\downarrow$
$a = -4$	*non-osc.* $\uparrow$	*osc.* $-$
$a < -4$	*non-osc.* $\uparrow$	*osc.* $\uparrow$

TABLE 2

	$y_n^+(z)$	$y_n^-(z)$
$z \in \mathbf{C} \backslash (\mathbf{R} \cup i\mathbf{R})$		
in $\mathscr{N}$	*osc.* $\uparrow$	*osc.* $\uparrow$
on $\mathscr{N}$	*osc.* $\uparrow$	*osc.* $-$
out $\mathscr{N}$	*osc.* $\uparrow$	*osc.* $\downarrow$
$z \in \mathbf{R}$, $\lvert z \rvert < 1$	*osc.* $\uparrow$	*osc.* $\uparrow$
$z \in \mathbf{R}$, $\lvert z \rvert > 1$	*non-osc.* $\uparrow$	*non-osc.* $\downarrow$
$z \in i\mathbf{R}$, $\lvert z \rvert > 1/\sqrt{3}$	*non-osc.* $\uparrow$	*osc.* $\downarrow$
$z = \pm i/\sqrt{3}$	*non-osc.* $\uparrow$	*osc.* $-$
$z \in i\mathbf{R}$, $\lvert z \rvert > 1/\sqrt{3}$	*non-osc.* $\uparrow$	*osc.* $\uparrow$

TABLE 3

	$Y_n^+(z)$	$Y_n^-(z)$
$z \in \mathbf{C} \backslash (-\infty, 1]$		
in $\mathscr{E}$	*osc.* $\downarrow$	*osc.* $\downarrow$
on $\mathscr{E}$	*osc.* $-$	*osc.* $\downarrow$
out $\mathscr{E}$	*osc.* $\uparrow$	*osc.* $\downarrow$
$z \in \mathbf{R}$, $\lvert z \rvert < 1$	*osc.* $\downarrow$	*osc.* $\downarrow$
$z \in \mathbf{R}$, $z < -1$		
in $\mathscr{E}$	*non-osc.* $\downarrow$	*non-osc.* $\downarrow$
on $\mathscr{E}$	*non-osc.* $-$	*non-osc.* $\downarrow$
out $\mathscr{E}$	*non-osc.* $\uparrow$	*non-osc.* $\downarrow$

In Table 1, $\mathscr{C}$ denotes the cardioid in the complex a-plane given by (34) (see Fig. 1). In Table 2, $\mathscr{N}$ represents the closed “walnut-shaped” curve obtained mapping the cardioid $\mathscr{C}$ into the complex z-plane by the transformation $z^2 = 1/(1+a)$ (see Fig. 2), in case when the coefficients of the difference equation depend holomorphically on z, cf. (37)–(39). Such a curve is symmetric with respect to both axes, and intersects the real axis in $z = \pm 1$, and the imaginary axis in $z = \pm i/\sqrt{3}$; observe that the inside [outside] of $\mathscr{C}$ is mapped into the outside [inside] of $\mathscr{N}$. In Table 3, $\mathscr{E}$ denotes the ellipse with foci in $z = \pm 1$ and semiaxes 3/4 and 5/4, on which $|\alpha_n(z)(\lambda_+(z))^n| = 1$, cf. (36) with $\alpha_n = \alpha_n(z) = \prod_{k=1}^{n-2}(z - a_k)/2$ (see also (4) and (37)).

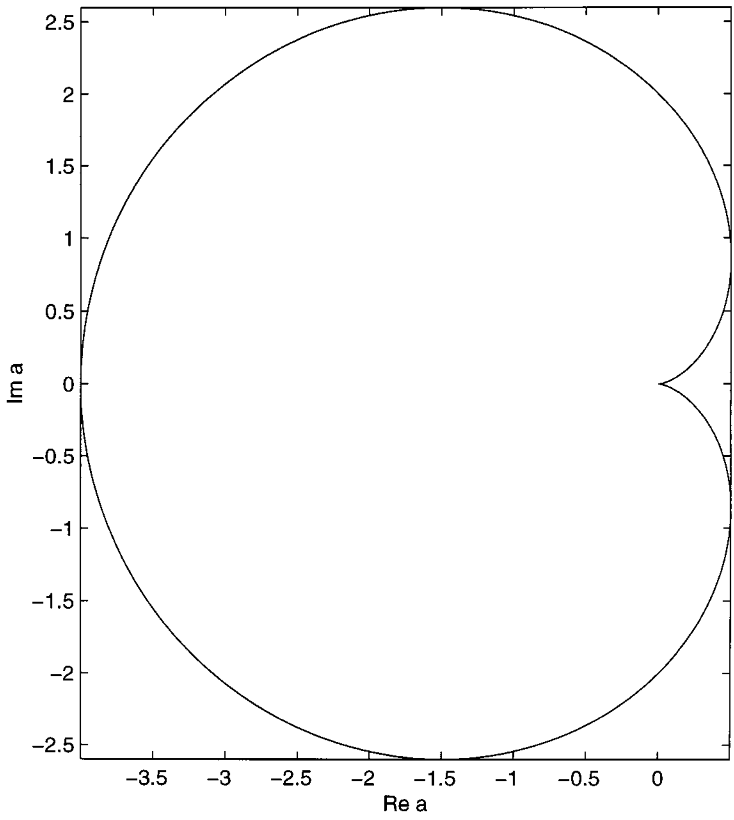

FIG. 1. The cardioid $\mathscr{C}$ defined by $r = 2(1 - \cos\theta)$, in the plane $a = re^{i\theta}$.

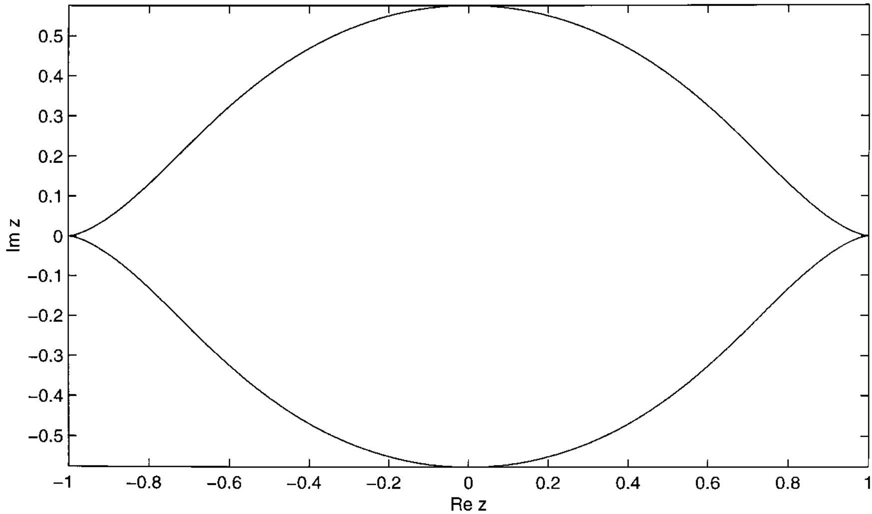

FIG. 2. The "walnut-shaped" curve $\mathcal{N}$ obtained from the cardioid $\mathcal{C}$ under the transformation $z^2 = 1/(1+a)$.

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